

49103014
ADDITIONAL COPY

NATIONAL BUREAU OF STANDARDS REPORT

AFOSR 1480

7345

INTERMEDIARY EQUATORIAL ORBITS OF AN ARTIFICIAL SATELLITE

by

John P. Vinti

PROPERTY OF
SOUTHWEST RESEARCH INSTITUTE LIBRARY
SAN ANTONIO, TEXAS



U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to government agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various consultation and information services. Research projects are also performed for other government agencies when the work relates to and supplements the basic program of the Bureau or when the Bureau's unique competence is required. The scope of activities is suggested by the listing of divisions and sections on the inside of the back cover.

Publications

The results of the Bureau's research are published either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three periodicals available from the Government Printing Office: The Journal of Research, published in four separate sections, presents complete scientific and technical papers; the Technical News Bulletin presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of non-periodical publications: Monographs, Applied Mathematics Series, Handbooks, Miscellaneous Publications, and Technical Notes.

A complete listing of the Bureau's publications can be found in National Bureau of Standards Circular 460, Publications of the National Bureau of Standards, 1901 to June 1947 (\$1.25), and the Supplement to National Bureau of Standards Circular 460, July 1947 to June 1957 (\$1.50), and Miscellaneous Publication 240, July 1957 to June 1960 (Includes Titles of Papers Published in Outside Journals 1950 to 1959) (\$2.25); available from the Superintendent of Documents, Government Printing Office, Washington 25, D. C.

NBS REPORT

7345

by

¹This work was supported by the U.S. Air Force, through the Office of Scientific Research of the Air Research and Development Command.

NATIONAL BUREAU OF STANDARDS Approved for public release by the
intended for use within the Govern Director of the National Institute of
to additional evaluation and review. Standards and Technology (NIST) on
listing of this Report, either in who October 9, 2015.
the Office of the Director, National
however, by the Government agency
to reproduce additional copies for

ss accounting documents
published it is subjected
duction, or open-literature
obtained in writing from
permission is not needed,
red if that agency wishes



**U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS**

INTERMEDIARY EQUATORIAL ORBITS OF AN ARTIFICIAL SATELLITE¹

by

John P. Vinti

A previous paper derived the solution for the drag-free motion of an artificial satellite in the gravitational field of an oblate planet. The corresponding potential, expressed in oblate spheroidal coordinates, leads to separability and represents the even zonal harmonics exactly through the second, for any oblate planet, and approximately through the fourth, in the case of the earth.

The previous paper contained a restriction on the orbital inclination I , viz., $I_c < I < 180^\circ - I_c$, where I_c might be as large as $1^\circ 54'$ for an orbit sufficiently close to the earth. The present paper removes this restriction and shows that many of the formulae for the periodic terms may be simplified, when the orbit is equatorial or almost so. The results agree with those obtained by a direct two-dimensional solution, when the orbit is purely equatorial.

1. This work was supported by the U.S. Air Force, through the Office of Scientific Research of the Air Research and Development Command.

1. Introduction

This paper is a sequel to a recent paper², concerning an accurate intermediary orbit for satellite astronomy, and will accordingly follow the notation thereof. It there followed that if

$$\lambda \equiv b_1/b_2 < 1, \quad (1)$$

all the ρ -integrals are expressible in terms of rapidly converging series involving products of Legendre polynomials with arguments λ and $(1-e^2)^{-\frac{1}{2}}$. Condition (1) is equivalent to a restriction on the orbital inclination I , viz.,

$$I_c < I < 180^\circ - I_c, \quad (2)$$

where, to the first order in k ,

$$\tan^2 I_c = k \equiv (r_e/p)^2 J_2 \quad (3)$$

For the earth $J_2 = 0.00108$, so that for orbits so close that $p \approx r_e$,

$$I_c = 1^\circ 54' \quad (4)$$

I imposed the condition $\lambda < 1$ in order that

$$(1 + A\rho^{-1} + B\rho^{-2})^{-\frac{1}{2}} \equiv (1 - 2\lambda h + h^2)^{-\frac{1}{2}} \quad (5)$$

should be a generating function for the Legendre polynomials $P_n(\lambda)$.

It now appears that such a restriction is unnecessary. Thus

$$(1 - 2\lambda h + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(\lambda) \quad (6)$$

2. J.P. Vinti, J. Research National Bureau of Standards, 65B,169-201,(1961), hereafter referred to as(A). Any reference in the present paper to an equation with a decimal number, such as(5.30), denotes an equation in (A).

even when $\lambda \geq 1$, provided only that⁽³⁾

$$h < \lambda - (\lambda^2 - 1)^{\frac{1}{2}} \quad (7)$$

We need show only that (7) is always satisfied, in order that all the results of (A) hold for all orbital inclinations. The only changes will be a few simplifications, especially for the cases $I = 0^\circ$ or 180° of purely equatorial orbits.

2. The ρ - Integrals

To show that (7) is always satisfied when $\lambda \geq 1$, note that

$$\frac{b_1}{\rho} = \frac{b_1}{b_2} \frac{b_2}{\rho} = \lambda h \quad (8)$$

By (4.12), (4.13), and (5.6) it follows that b_1 and b_2 are both real and non-negative for $c^2 < ap$, a relation that always holds for satellite orbits. Thus (7) is satisfied if and only if

$$b_1/\rho < g(\lambda), \quad (9)$$

where

$$g(\lambda) \equiv \lambda^2 - \lambda(\lambda^2 - 1)^{\frac{1}{2}} \quad (10)$$

We shall next show that

$$1/2 < g(\lambda) \leq 1 \quad (\lambda \geq 1) \quad (11)$$

To show that $g(\lambda) \leq 1$, note that for $\lambda \geq 1$, we have $1 > 1 - \lambda^{-2} \geq 0$, so that $(1 - \lambda^{-2})^{\frac{1}{2}} \geq 1 - \lambda^{-2}$, $1 - (1 - \lambda^{-2})^{\frac{1}{2}} \leq \lambda^{-2}$, and finally $\lambda^2 - \lambda(\lambda^2 - 1)^{\frac{1}{2}} \leq 1$.

Thus $g(\lambda) \leq 1$.

To show that $g(\lambda) > 1/2$, note that for $\lambda \geq 1$

$$0 \leq 1 - \lambda^{-2} < (1 - \frac{1}{2} \lambda^{-2})^2, \quad (12)$$

3. E.W. Hobson, "The Theory of Spherical and Ellipsoidal Harmonics", page 15, Cambridge University Press, Cambridge 1931.

so that

$$(1-\lambda^{-2})^{\frac{1}{2}} < 1 - \frac{1}{2} \lambda^{-2} \quad (13)$$

or

$$1 - (1-\lambda^{-2})^{\frac{1}{2}} > \frac{1}{2} \lambda^{-2} \quad (14)$$

Then

$$g(\lambda) \equiv \lambda^2 - \lambda(\lambda^2 - 1)^{\frac{1}{2}} > \frac{1}{2}, \quad (15)$$

as was to be shown.

From (9) and (11) it follows that when $\lambda \geq 1$ the condition $b_1/\rho < 1$ is necessary for the validity of (7) and that the condition $b_1/\rho < 1/2$ is sufficient for its validity.

To show that $b_1/\rho < 1/2$ for all bound⁽⁴⁾ orbits, note that $\rho \geq \rho_1$, so that

$$b_1/\rho \leq b_1/\rho_1 \quad (16)$$

From the relation $c^2 = kp^2$ and the relation $\rho_1 = a(1-e)$, $e \leq 1$, for a bound orbit, we then find from (3.25), (4.12), and (5.6) that

$$\frac{b_1}{\rho_1} = \frac{k(1+e)(1-\gamma_0^2)[1-k\gamma_0^2(1-e^2)]}{[1-k(1-e^2)][1-k\gamma_0^2(1-e^2)] + 4k\gamma_0^2} \quad (17)$$

Here the numerator $\leq k(1+e)$ and the denominator $\geq (1-k)^2$, so that

$$\frac{b_1}{\rho_1} \leq \frac{k(1+e)}{(1-k)^2} \leq \frac{2k}{(1-k)^2}, \quad (18)$$

a function monotonic in k for $0 < k < 1$.

For the earth $k < 0.00108$, so that

$$b_1/\rho_1 < 0.00216 < \frac{1}{2} \quad (19)$$

4. Past participle of the verb "to bind", taken from the terminology of atomic theory.

From (16) and (19) it follows that for the earth

$$b_1/\rho < 1/2 \quad (20)$$

This is the condition sufficient for the validity of (7) and thus of the expansion (6), for all $\lambda \geq 1$.

All the developments of (A) then hold for the ρ -integrals, in particular (5.30) through (5.33) for the integral R_1 , (5.35) through (5.40) for R_2 , and (5.60) through (5.65) for R_3 , where the D_m 's are again given by (5.50) and (5.53).

To show the rapid convergence of the various series that there occur, we note first that since

$$P_n(\lambda) = \pi^{-1} \int_0^\pi [\lambda + (\lambda^2 - 1)^{\frac{1}{2}} \cos x]^n dx \quad (21)$$

and since $\lambda \equiv b_1/b_2 \geq 1$, we have

$$(b_2/p)^n P_n(\lambda) = \pi^{-1} \int_0^\pi [b_1 p^{-1} + p^{-1} (b_1^2 - b_2^2)^{\frac{1}{2}} \cos x]^n dx \quad (22)$$

Then

$$\left| \left(\frac{b_2}{p} \right)^n P_n(\lambda) \right| \leq \left[\frac{b_1}{p} + \frac{b_1}{p} (1 - \lambda^{-2})^{\frac{1}{2}} \right]^n \leq \left(\frac{2b_1}{p} \right)^n \quad (23)$$

From (5.14) and (5.34) the series S_1 and S_2 that occur in the expressions for the ρ -integrals R_1 and R_2 are

$$S_j \equiv \sum_{n=n_j}^{\infty} \left(\frac{b_2}{p} \right)^n P_n(\lambda) \int_0^v (1 + e \cos x)^{n-n_j} dx, \quad (j=1,2) \quad (24)$$

where $n_1 = 2$ and $n_2 = 0$. Thus

$$|S_j| \leq v \sum_{n=n_j}^{\infty} \left(\frac{2b_1}{p} \right)^n (1+e)^{n-n_j} \quad (25)$$

$$\leq v \sum_{n=0}^{\infty} \left(\frac{2b_1}{p} \right)^{n+n_j} (1+e)^n \leq v \left(\frac{2b_1}{p} \right)^{n_j} \sum_{n=0}^{\infty} \left[\frac{2b_1}{p} (1+e) \right]^n \quad (26)$$

$$\leq \frac{v \left(\frac{2b_1}{p} \right)^{n_j}}{1 - \frac{2b_1}{p} (1+e)} \quad (j = 1, 2) \quad (27)$$

But $p = a(1-e^2) = \rho_1(1+e)$, so that by (18)

$$b_1 p^{-1} \leq k(1-k)^{-2} \quad (28)$$

and

$$2b_1(1+e) p^{-1} \leq 2k(1+e)(1-k)^{-2} \leq 4k(1-k)^{-2}, \quad (29)$$

where $4k(1-k)^{-2} < 0.0043$ for the earth. It follows that the series for R_1 and R_2 , and thus the series for the secular coefficients A_1 and A_2 , converge absolutely and more rapidly than a geometric series of common ratio 0.0043.

By (5.49), (5.50), and (5.53) the series S_3 that occurs in the expression for the ρ -integral R_3 is

$$S_3 = S_e + S_o, \quad (30)$$

$$S_e = \sum_{n=0}^{\infty} D_{2n} \int_0^v (1 + e \cos x)^{2n+2} dx \quad (31)$$

$$S_o \equiv \sum_{n=0}^{\infty} D_{2n+1} \int_0^v (1 + e \cos x)^{2n+3} dx \quad (32)$$

$$D_{2n} \equiv \sum_{j=0}^n (-1)^{n-j} (c/p)^{2n-2j} (b_2/p)^{2j} P_{2j}(\lambda) \quad (33)$$

$$D_{2n+1} \equiv \sum_{j=0}^n (-1)^{n-j} (c/p)^{2n-2j} (b_2/p)^{2j+1} P_{2j+1}(\lambda) \quad (34)$$

From (33) and (23) and the relation $c^2 = kp^2$, it follows that

$$\begin{aligned} |D_{2n}| &\leq k^n \sum_{j=0}^n (2b_1/c)^{2j} \leq k^n \sum_{j=0}^{\infty} (2b_1/c)^{2j} \\ &= \frac{k^n}{1 - (2b_1/c)^2} \end{aligned} \quad (35)$$

and from (34) and (23) that

$$\begin{aligned} |D_{2n+1}| &\leq k^n (2b_1/p) \sum_{j=0}^n (2b_1/c)^{2j} \leq k^n (2b_1/p) \sum_{j=0}^{\infty} (2b_1/c)^{2j} \\ &= \frac{k^n (2b_1/p)}{1 - (2b_1/c)^2} \end{aligned} \quad (36)$$

Thus, by (31) and (35)

$$\begin{aligned} |S_e| &\leq \frac{(1+e)^2 v}{1 - (2b_1/c)^2} \sum_{n=0}^{\infty} [k(1+e)^2]^n \\ &= \frac{(1+e)^2 v}{[1 - (2b_1/c)^2][1 - k(1+e)^2]} \end{aligned} \quad (37)$$

and by (32) and (36) that

$$\begin{aligned}
 |S_0| &\leq \frac{(1+e)^3 v (2b_1/p)}{[1 - (2b_1/c)^2]} \sum_{n=0}^{\infty} [k(1+e)^2]^n \\
 &\leq \frac{(1+e)^3 v (2b_1/p)}{[1 - (2b_1/c)^2][1 - k(1+e)^2]} \quad (38)
 \end{aligned}$$

Then, by (30), (37), and (38), we have

$$|S_3| \leq \frac{(1+e)^2 v [1 + (1+e)(2b_1/p)]}{[1 - (2b_1/c)^2][1 - k(1+e)^2]} \quad (39)$$

But $p = \rho_1(1+e)$, so that by (18)

$$2b_1/p \leq 2k(1-k)^{-2} \quad (40)$$

and

$$2b_1/c \equiv (2b_1/p) k^{-\frac{1}{2}} \leq 2k^{\frac{1}{2}}(1-k)^{-2} \quad (41)$$

Thus

$$|S_3| \leq \frac{(1+e)^2 v [1 + 2k(1+e)(1-k)^{-2}]}{[1 - 4k(1-k)^{-4}][1 - k(1+e)^2]} \quad (42)$$

The series for S_3 thus converges absolutely and more rapidly than the power series in k of the function on the right side of (42), where $k < 0.0011$ for the earth. On replacing v by π , we can then say the same thing about the secular coefficient A_3 .

3. Simplification of the ρ -Coefficients when $\lambda \geq 1$

From $\lambda \equiv b_1/b_2$ it follows that

$$b_2 = b_1 \lambda^{-1} \leq b_1 \quad (\lambda \geq 1) \quad (43)$$

Since $b_1=O(k)$, it then follows that b_2 is also of order k when $\lambda \geq 1$. This fact enables us to simplify the coefficients A_{1n} and A_{2n} , which are needed only to $O(k^2)$, and the coefficients A_{3n} , which are needed only to $O(k)$.

Thus (5.32) and (5.33) lead to

$$A_{11} = O(k^3) \quad A_{12} = O(k^4), \quad (44)$$

(5.39) and (5.40) lead to

$$A_{23} = O(k^3) \quad A_{24} = O(k^4), \quad (45)$$

and (5.37) and (5.38) lead to

$$A_{21} = (1-e^2)^{\frac{1}{2}} p^{-1} e [b_1 p^{-1} + (3-\lambda^{-2}) k^2 \cos^4 I] \quad (46)$$

$$A_{22} = (1-e^2)^{\frac{1}{2}} p^{-1} \frac{e^2}{8} (3-\lambda^{-2}) k^2 \cos^4 I \quad (47)$$

Here we have used

$$b_1 p^{-1} = k \cos^2 I + O(k^2) \quad (48)$$

in the terms involving $b_1^2 p^{-2}$.

Similarly, for the coefficients A_{3n} , which are needed only to $O(k)$, we obtain from (5.62) through (5.65)

$$A_{31} = (1-e^2)^{\frac{1}{2}} p^{-3} e [2 + (3 + \frac{3}{4} e^2) k \cos^2 I - (4 + 3e^2)k] \quad (49)$$

$$A_{32} = (1-e^2)^{\frac{1}{2}} p^{-3} e^2 [\frac{1}{4} + \frac{3}{4} k \cos^2 I - (\frac{e^2}{4} + \frac{3}{2})k] \quad (50)$$

$$A_{33} = (1-e^2)^{\frac{1}{2}} p^{-3} e^3 [\frac{k}{12} \cos^2 I - \frac{k}{3}] \quad (51)$$

$$A_{34} = - (1-e^2)^{\frac{1}{2}} p^{-3} e^4 k/32 \quad (52)$$

For $\lambda \geq 1$ we need also to rewrite (5.31), (5.36), (5.50), and (5.53) for the secular coefficients A_1 , A_2 , D_{2n} , and D_{2n+1} , which all contain terms of the form $(b_2/p)^m P_m(\lambda)$. As $\sin I$ approaches zero, b_2 also approaches zero and λ becomes infinite, so that such a term takes the indeterminate form zero times infinity. To remove this indeterminacy, note that

$$\frac{b_2}{p} \equiv \frac{b_2}{b_1} \frac{b_1}{p} \equiv \frac{b_1}{p} \lambda^{-1}, \quad (53)$$

so that

$$\left(\frac{b_2}{p}\right)^m P_m(\lambda) = \left(\frac{b_1}{p}\right)^m \lambda^{-m} P_m(\lambda) \quad (54)$$

$$= \left(\frac{b_1}{p}\right)^m R_m(\lambda^{-1}) \quad (55)$$

Here $R_m(x)$ is a function that has already appeared in (A), viz.,

$$R_m(x) \equiv x^m P_m(x^{-1}) \quad (56)$$

a polynomial of degree $[m/2]$ in x^2 .

To determine $(b_2/p)^m P_m(\lambda)$ for $\sin I = 0$, first write

$$P_m(\lambda) = \sum_{j=0}^{[m/2]} \frac{(-1)^j (2m-2j)! \lambda^{m-2j}}{2^m j! (m-j)! (m-2j)!}, \quad (57)$$

so that

$$\begin{aligned} R_m(\lambda^{-1}) &\equiv \lambda^{-m} P_m(\lambda) = 1 \quad (m=0,1) \\ &= \frac{(2m)!}{2^m (m!)^2} + \sum_{j=1}^{[m/2]} \frac{(-1)^j (2m-2j)! \lambda^{-2j}}{2^m j! (m-j)! (m-2j)!} \quad (m=2,3,4,\dots) \end{aligned} \quad (58)$$

Thus

$$R_m(0) = \frac{(2m)!}{2^m (m!)^2}, \quad (m = 0, 1, 2, 3, \dots) \quad (59)$$

so that by (55) and (59)

$$(b_2/p)^m P_m(\lambda) = \frac{(2m)!}{(m!)^2} \left(\frac{b_{10}}{2p} \right)^m, \quad (\sin I = 0) \quad (60)$$

where b_{10} is the value of b_1 for $\gamma_0 \equiv \sin I = 0$.

From (5.31), (5.36), (55), and (60) it then follows that

$$A_1 = (1-e^2)^{\frac{1}{2}p} \sum_{n=2}^{\infty} (b_1/p)^n R_n(\lambda^{-1}) R_{n-2}(\sqrt{1-e^2}) \quad (\lambda \geq 1) \quad (61)$$

$$= (1-e^2)^{\frac{1}{2}p} \sum_{n=2}^{\infty} \frac{(2n)!}{(n!)^2} \left(\frac{b_{10}}{2p} \right)^n R_{n-2}(\sqrt{1-e^2}) \quad (\sin I = 0) \quad (62)$$

$$A_2 = (1-e^2)^{\frac{1}{2}p-1} \sum_{n=0}^{\infty} (b_1/p)^n R_n(\lambda^{-1}) R_n(\sqrt{1-e^2}) \quad (\lambda \geq 1) \quad (63)$$

$$= (1-e^2)^{\frac{1}{2}p-1} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \left(\frac{b_{10}}{2p} \right)^n R_n(\sqrt{1-e^2}) \quad (\sin I = 0) \quad (64)$$

Note that $A_1 = O(k^2)$ and $A_2 = O(k^0)$.

Then, from (5.61)

$$A_3 = (1-e^2)^{\frac{1}{2}p-3} \sum_{m=0}^{\infty} D_m R_{m+2}(\sqrt{1-e^2}), \quad (65)$$

where, by (5.50), (5.53), (55), (60), and the relation $c^2 = kp^2$, we find

$$\begin{aligned}
 D_{2n} &= \sum_{j=0}^n (-1)^{n-j} (c/p)^{2n-2j} (b_2/p)^{2j} P_{2j}(\lambda) \\
 &= \sum_{j=0}^n (-1)^{n-j} k^{n-j} (b_1/p)^{2j} R_{2j}(\lambda^{-1}) \quad (\lambda \geq 1) \quad (66)
 \end{aligned}$$

$$= \sum_{j=0}^n (-1)^{n-j} k^{n-j} \frac{(4j)!}{[(2j)!]^2} \left(-\frac{b_{10}}{2p}\right)^{2j} \quad (\sin I = 0) \quad (67)$$

$$\begin{aligned}
 D_{2n+1} &= \sum_{j=0}^n (-1)^{n-j} (c/p)^{2n-2j} (b_2/p)^{2j+1} P_{2j+1}(\lambda) \\
 &= \sum_{j=0}^n (-1)^{n-j} k^{n-j} (b_1/p)^{2j+1} R_{2j+1}(\lambda^{-1}) \quad (\lambda \geq 1) \quad (68)
 \end{aligned}$$

$$\begin{aligned}
 &\pi - \overbrace{(n-1)} \\
 &= \sum_{j=0}^n (-1)^{n-j} k^{n-j} \frac{(4j+2)!}{[(2j+1)!]^2} \left(-\frac{b_{10}}{2p}\right)^{2j+1} \quad (\sin I = 0) \quad (69)
 \end{aligned}$$

4. Simplification of Other Coefficients when $\lambda \geq 1$

To simplify the η -coefficients when $\lambda \geq 1$, we must first note that from page (176) of (A) we have

$$b_1 \approx k p \cos^2 I \quad b_2 \approx k^{\frac{1}{2}} p \sin I, \quad (70)$$

so that

$$\lambda \equiv b_1/b_2 \approx k^{\frac{1}{2}} \cos^2 I \csc I \quad (71)$$

or

$$\eta_0 \equiv \sin I \approx k^{\frac{1}{2}} \lambda^{-1} \cos^2 I \quad (72)$$

Thus

$$\eta_0 = O(k^{\frac{1}{2}}) \quad (\lambda \geq 1) \quad (73)$$

Also, since $\eta_2^{-1} = O(k^{\frac{1}{2}})$, by (3.42), it follows that

$$q \equiv \eta_0/\eta_2 = O(k) \quad (\lambda \geq 1) \quad (74)$$

There also arises an indeterminacy in the η -coefficients, from the quantity $(\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0$, which takes the form infinity times zero as

$\eta_0 \equiv \sin I$ approaches zero. To remove this indeterminacy, use (4.15a), viz.,

$$\alpha_3 = \alpha_2 \left(1 - \frac{c^2 \eta_0^2}{a_0 p_0} \right)^{\frac{1}{2}} \cos I, \quad (75)$$

to find

$$(\alpha_2^2 - \alpha_3^2)^{-\frac{1}{2}} \eta_0 = \alpha_2^{-1} \left(1 + \frac{c^2}{a_0 p_0} \cos^2 I \right)^{-\frac{1}{2}} \quad (76)$$

Then, from (8.27) and (7.23)

$$\psi_s = 2\pi \nu_2 [t + \beta_1 + \beta_2 \alpha_2^{-1} (a + b_1 + A_1) A_2^{-1}] \quad (77)$$

$$2\pi \nu_2 = \alpha_2 \left(1 + \frac{c^2}{a_0 p_0} \cos^2 I \right)^{\frac{1}{2}} A_2 B_2^{-1} (a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1})^{-1} \quad (78)$$

From (8.33)

$$\psi_0 = (-2\alpha_1)^{-\frac{1}{2}} \alpha_2 \left(1 + \frac{c^2}{a_0 p_0} \cos^2 I \right)^{\frac{1}{2}} A_2 B_2^{-1} v_0 \quad (79)$$

From (8.37)

$$M_1 = (a + b_1)^{-1} [-(A_1 + c^2 \gamma_0^2 A_2 B_1 B_2^{-1}) v_0 + \frac{c^2}{4} (-2\alpha_1)^{\frac{1}{2}} \alpha_2^{-1} (1 + \frac{c^2}{a_0 p_0} \cos^2 I)^{-\frac{1}{2}} \gamma_0^2 \sin(2\psi_s + 2\psi_0)], \quad (80)$$

of order k^2 . Then, by (8.39)

$$E_1 = [1 - e' \cos(M_s + E_0)]^{-1} M_1, \quad (81)$$

since $M_1^2 = O(k^4)$. Then E_1 and v_1 are both of order k^2 . Also, by (8.40),

$$\psi_1 = (-2\alpha_1)^{-\frac{1}{2}} \alpha_2 (1 + \frac{c^2}{a_0 p_0} \cos^2 I)^{\frac{1}{2}} B_2^{-1} [A_2 v_1 + A_{21} \sin(M_s + v_0) + A_{22} \sin(2M_s + 2v_0)] + \frac{q}{8} B_2^{-1} \sin(2\psi_s + 2\psi_0) \quad (82)$$

By (8.45), M_2 and thus E_2 and v_2 are of order k^3 and, by (8.48), ψ_2 is also of order k^3 . Thus all the second-order periodic terms of (A) become of the third order and thus negligible, when $\lambda \geq 1$.

By (8.50) the right ascension ϕ becomes

$$\phi = \beta_3 + \alpha_3 \alpha_2^{-1} (1 + \frac{c^2}{a_0 p_0} \cos^2 I)^{-\frac{1}{2}} [(1 - \gamma_0^2)^{-\frac{1}{2}} (1 - \gamma_2^{-2})^{-\frac{1}{2}} \chi + B_3 \psi] - c^2 \alpha_3 (-2\alpha_1)^{-\frac{1}{2}} (A_3 v + \sum_{n=1}^4 A_{3n} \sin nv) \quad (83)$$

Here we have dropped the periodic term $(3/32) \gamma_0^2 \gamma_2^{-4} \sin 2\psi$ of (8.50), since it is of order k^3 for $\lambda \geq 1$.

5. Summary for $\lambda \geq 1$

For an almost equatorial orbit, corresponding to $\lambda \geq 1$, the right ascension ϕ is given by (83). The spheroidal coordinates ρ and η are given by

$$\rho = a(1 - e \cos E) = (1 + e \cos v)^{-1} p$$

$$\eta = \eta_0 \sin \psi$$

Here the expressions

$$E = M_s + E_0 + E_1 \quad v = M_s + v_0 + v_1 \quad \psi = \psi_s + \psi_0 + \psi_1$$

are sufficiently accurate to give the secular terms exactly and the periodic terms correctly through order k^2 , provided that M_s is calculated by (8.24), ψ_s by (77), E_0 by (8.31), E_1 by (80) and (81), v_0 and v_1 by the anomaly relations (8.1), ψ_0 by (79), and ψ_1 by (82).

6. The Case of a Purely Equatorial Orbit, $I = 0^\circ$ or 180°

For $I = 0^\circ$ or 180° we have $\eta_0 = 0$, $\cos^2 I = 1$, $\chi = \psi$ by (6.51), $\gamma_m = 0$ by (6.66),

$$B_3 = 1 - (1 - \eta_2^{-2})^{-\frac{1}{2}} \quad (84)$$

by (6.65), and $|\alpha_3| = \alpha_2$, so that

$$\alpha_3 / \alpha_2 = \operatorname{sgn} \alpha_3 \quad (85)$$

Then, by (83),

$$\phi = \beta_3 + (\operatorname{sgn} \alpha_3) \left(1 + \frac{c^2}{a_0 p_0}\right)^{-\frac{1}{2}} \psi - c^2 \alpha_3 (-2\alpha_1)^{-\frac{1}{2}} (A_3 v + \sum_{n=1}^4 A_{3n} \sin nv) \quad (86)$$

Also, by (8.24) with $\eta_0 = 0$,

$$M_s = (-2\alpha_1)^{\frac{1}{2}} (a + b_1 + A_1)^{-1} (t + \beta_1) \quad (87)$$

and by (77) and (78), with $\eta_0 = 0$, $\cos^2 I = 1$, and $B_2 = 1$,

$$\psi_s = \beta_2 \left(1 + \frac{c^2}{a_0 p_0}\right)^{\frac{1}{2}} + \alpha_2 A_2 \left(1 + \frac{c^2}{a_0 p_0}\right)^{\frac{1}{2}} (a + b_1 + A_1)^{-1} (t + \beta_1) \quad (88)$$

Thus, by (87) and (88),

$$\psi_s = \beta_2 \left(1 + \frac{c^2}{a_0 p_0}\right)^{\frac{1}{2}} + \alpha_2 \left(1 + \frac{c^2}{a_0 p_0}\right)^{\frac{1}{2}} (-2\alpha_1)^{-\frac{1}{2}} A_2 M_s \quad (89)$$

Then, by (79) and (82), with $\eta_0 = 0$,

$$\psi_0 = \alpha_2 \left(1 + \frac{c^2}{a_0 p_0}\right)^{\frac{1}{2}} (-2\alpha_1)^{-\frac{1}{2}} A_2 v_0 \quad (90)$$

$$\psi_1 = \alpha_2 \left(1 + \frac{c^2}{a_0 p_0}\right)^{\frac{1}{2}} (-2\alpha_1)^{-\frac{1}{2}} \left[A_2 v_1 + \sum_{n=1}^2 A_{2n} \sin(n M_s + n v_0) \right] \quad (91)$$

Addition of (89) through (91) then gives

$$\psi = \beta_2 \left(1 + \frac{c^2}{a_0 p_0}\right)^{\frac{1}{2}} + \alpha_2 \left(1 + \frac{c^2}{a_0 p_0}\right)^{\frac{1}{2}} (-2\alpha_1)^{-\frac{1}{2}} \left[A_2 v + \sum_{n=1}^2 A_{2n} \sin(n M_s + n v_0) \right] \quad (92)$$

Since $v = M_s + v_0 + v_1$, where v_1 is of order k^2 , and since A_{21} and A_{22} are of orders k and k^2 respectively, it follows that

$$\sum_{n=1}^2 A_{2n} \sin(n M_s + n v_0) = \sum_{n=1}^2 A_{2n} \sin n v \quad (93)$$

to order k^2 . Thus, to order k^2 ,

$$\psi = \beta_2 \left(1 + \frac{c^2}{a_0 p_0}\right)^{\frac{1}{2}} + \alpha_2 \left(1 + \frac{c^2}{a_0 p_0}\right)^{\frac{1}{2}} (-2\alpha_1)^{-\frac{1}{2}} (A_2 v + \sum_{n=1}^2 A_{2n} \sin n v) \quad (94)$$

On inserting (94) into (86), we then find

$$\begin{aligned} \phi = & \beta_3 + \beta_2 \operatorname{sgn} \alpha_3 + \alpha_3 (-2\alpha_1)^{-\frac{1}{2}} (A_2 v + \sum_{n=1}^2 A_{2n} \sin nv) \\ & - c^2 \alpha_3 (-2\alpha_1)^{-\frac{1}{2}} (A_3 v + \sum_{n=1}^4 A_{3n} \sin nv) \end{aligned} \quad (95)$$

It is a simple exercise to check these results for a purely equatorial orbit. To do so, let X and Y be the usual Cartesian coordinates, define ρ and ϕ by

$$X + iY = (\rho^2 + c^2)^{\frac{1}{2}} \exp i\phi, \quad (96)$$

write down the kinetic energy $\frac{1}{2}(\dot{X}^2 + \dot{Y}^2)$ in terms of ρ and ϕ and their time derivatives, write the potential as $-\mu\rho^{-1}$, construct the Hamiltonian, and then write down the Hamilton-Jacobi equation. Separate the latter, to obtain the solution

$$W = \alpha_3 \phi \pm \int_{\rho_1}^{\rho} (\rho^2 + c^2)^{-1} F(\rho)^{\frac{1}{2}} d\rho, \quad (97)$$

where

$$F(\rho) \equiv c^2 \alpha_3^2 + (\rho^2 + c^2) (-\alpha_3^2 + 2\mu\rho + 2\alpha_1 \rho^2) \quad (98)$$

The kinetic equations are then

$$\frac{\partial W}{\partial \alpha_1} = t + \beta_1 = \pm \int_{\rho_1}^{\rho} \rho^2 F(\rho)^{\frac{1}{2}} d\rho \quad (99)$$

$$\frac{\partial W}{\partial \alpha} = \beta'_3 = \phi + \alpha_3 \int_{\rho_1}^{\rho} (\rho^2 + c^2)^{-1} \rho^2 F(\rho)^{-\frac{1}{2}} d\rho \quad (100)$$

$$= \phi + \alpha_3 \int_{\rho_1}^{\rho} F(\rho)^{-\frac{1}{2}} d\rho \pm c^2 \alpha_3 \int_{\rho_1}^{\rho} (\rho^2 + c^2)^{-1} F(\rho)^{-\frac{1}{2}} d\rho \quad (101)$$

On then following the procedure in (A), we find that ρ is given by just the results of section 5 of the present paper, with τ_0 placed equal to zero, and that ϕ is given by

$$\begin{aligned} \phi = \beta'_3 + \alpha_3 (-2\alpha_1)^{-\frac{1}{2}} (A_2 v + \sum_{n=1}^2 A_{2n} \sin nv) \\ - c^2 \alpha_3 (-2\alpha_1)^{-\frac{1}{2}} (A_3 v + \sum_{n=1}^4 A_{3n} \sin nv) \end{aligned} \quad (102)$$

Comparison of (102) with (95) shows that the results agree if

$$\beta'_3 = \beta_3 + \beta_2 \operatorname{sgn} \alpha_3 \quad (103)$$

In the equatorial plane $r^2 = \rho^2 + c^2$, so that r is at minimum whenever ρ is at minimum. That is, the satellite is at perigee whenever $v = 2\pi \tau$, $\tau = 0, 1, 2, \dots$. By (102) the right ascension ϕ_p thus changes value from one perigee to the next. For an equatorial orbit about a planet of zero oblateness, however, the coefficient of v in (102), viz.,

$\alpha_3 (-2\alpha_1)^{-\frac{1}{2}} (A_2 - c^2 A_3)$, would reduce to $\operatorname{sgn} \alpha_3$, so that in such a limiting case

$$\phi_p = \beta'_3 + 2\pi \tau \operatorname{sgn} \alpha_3, \quad (\tau = 0, 1, 2, \dots) \quad (104)$$

so that the actual position of perigee would remain fixed and β'_3 would be its right ascension.

But for the case of a non-equatorial orbit around a planet of zero oblateness β_3 is simply the right ascension Ω of the ascending node and β_2 is the argument ω of perigee. Thus for the limiting case of a purely equatorial orbit about a planet of zero oblateness (103) would take the expected form

$$\text{R.A. of perigee} = \Omega \pm \omega, \quad (105)$$

where the sign would be plus for a direct orbit and minus for a retrograde orbit. Thus the result (103) is reasonable.

U. S. DEPARTMENT OF COMMERCE
Luther H. Hodges, *Secretary*

NATIONAL BUREAU OF STANDARDS
A. V. Astin, *Director*



THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards at its major laboratories in Washington, D.C., and Boulder, Colorado, is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section carries out specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant publications, appears on the inside of the front cover.

WASHINGTON, D.C.

Electricity. Resistance and Reactance. Electrochemistry. Electrical Instruments. Magnetic Measurements. Dielectrics.

Metrology. Photometry and Colorimetry. Refractometry. Photographic Research. Length. Engineering Metrology. Mass and Scale. Volumetry and Densimetry.

Heat. Temperature Physics. Heat Measurements. Cryogenic Physics. Equation of State. Statistical Physics. Radiation Physics. X-ray. Radioactivity. Radiation Theory. High Energy Radiation. Radiological Equipment. Nucleonic Instrumentation. Neutron Physics.

Analytical and Inorganic Chemistry. Pure Substances. Spectrochemistry. Solution Chemistry. Standard Reference Materials. Applied Analytical Research.

Mechanics. Sound. Pressure and Vacuum. Fluid Mechanics. Engineering Mechanics. Rheology. Combustion Controls.

Organic and Fibrous Materials. Rubber. Textiles. Paper. Leather. Testing and Specifications. Polymer Structure. Plastics. Dental Research.

Metallurgy. Thermal Metallurgy. Chemical Metallurgy. Mechanical Metallurgy. Corrosion. Metal Physics. Electrolysis and Metal Deposition.

Mineral Products. Engineering Ceramics. Glass. Refractories. Enameled Metals. Crystal Growth. Physical Properties. Constitution and Microstructure.

Building Research. Structural Engineering. Fire Research. Mechanical Systems. Organic Building Materials. Codes and Safety Standards. Heat Transfer. Inorganic Building Materials.

Applied Mathematics. Numerical Analysis. Computation. Statistical Engineering. Mathematical Physics. Operations Research.

Data Processing Systems. Components and Techniques. Digital Circuitry. Digital Systems. Analog Systems. Applications Engineering.

Atomic Physics. Spectroscopy. Infrared Spectroscopy. Solid State Physics. Electron Physics. Atomic Physics. Instrumentation. Engineering Electronics. Electron Devices. Electronic Instrumentation. Mechanical Instruments. Basic Instrumentation.

Physical Chemistry. Thermochemistry. Surface Chemistry. Organic Chemistry. Molecular Spectroscopy. Molecular Kinetics. Mass Spectrometry.

Office of Weights and Measures.

BOULDER, COLO.

Cryogenic Engineering. Cryogenic Equipment. Cryogenic Processes. Properties of Materials. Cryogenic Technical Services.

Ionosphere Research and Propagation. Low Frequency and Very Low Frequency Research. Ionosphere Research. Prediction Services. Sun-Earth Relationships. Field Engineering. Radio Warning Services.

Radio Propagation Engineering. Data Reduction Instrumentation. Radio Noise. Tropospheric Measurements. Tropospheric Analysis. Propagation-Terrain Effects. Radio-Meteorology. Lower Atmosphere Physics.

Radio Standards. High Frequency Electrical Standards. Radio Broadcast Service. Radio and Microwave Materials. Atomic Frequency and Time Interval Standards. Electronic Calibration Center. Millimeter-Wave Research. Microwave Circuit Standards.

Radio Systems. High Frequency and Very High Frequency Research. Modulation Research. Antenna Research. Navigation Systems.

Upper Atmosphere and Space Physics. Upper Atmosphere and Plasma Physics. Ionosphere and Exosphere Scatter. Airglow and Aurora. Ionospheric Radio Astronomy.

